INTEGRAL EXPANSION OF AN ARBITRARY FUNCTION IN TERMS OF SPHERICAL FUNCTIONS

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In this paper the authors obtain a theorem of expansion, related to the Mehler-Fok integral expansion, of an arbitrary function in form of an integral and in terms of spherical functions. Below, a method is given for solving boundary value problems encountered in mathematical physics and in the theory of elasticity for a hyperboloid of revolution of one sheet.

1. Introduction. In the process of solution of many problems of mathematical physics an important part is played by the representation of an arbitrary function defined on the interval $(1, \infty)$, in form of Mehler-Fok integral [1 to 3]

$$f(x) = \int_0^\infty \tau \tanh \pi\tau \, P_{-1/2+i\tau}(x)\, d\tau \int_1^\infty f(y)\, P_{-1/2+i\tau}(y)\, dy \qquad (1 < x < \infty) \quad (1.1)$$

where $P_\nu(x)$ is a spherical Legendre function of the first kind.

In particular, the above representation appears when the method of separation of variables is applied to harmonic boundary problems for the regions bounded by the surface of a hyperboloid of revolution of two sheets, or by the surface of intersection of two spheres [3 to 5], or to some problems in the theory of elasticity [6].

In this paper we shall consider the following integral expansion

$$f(x) = \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh \pi\tau} \left\{ \frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} \int_{-\infty}^{\infty} f(y) \frac{P_{-1/2+i\tau}(iy) + P_{-1/2+i\tau}(-iy)}{2} dy + \right.$$

$$\left. + \frac{P_{-1/2+i\tau}(ix) - P_{-1/2+i\tau}(-ix)}{2i} \int_{-\infty}^{\infty} f(y) \frac{P_{-1/2+i\tau}(iy) - P_{-1/2+i\tau}(-iy)}{2i} dy \right\} d\tau$$

$$(-\infty < x < \infty) \qquad (1.2)$$

which represents an analog of (1.1) for the boundary value problems existing in the regions bounded by the surface of a hyperboloid of revolution of one sheet.

For a narrower class of functions, the representation (1.2) can also be obtained from the general theory of expansion of the linear, singular differential operators in terms of their characteristic functions ([7] Section 2, pp.454-455 and [8], pp.119-120).

The aim of this paper is to give a direct and relatively simple proof of Formula (1.2) which would be based on direct investigation of the properties of spherical functions and which would establish the validity of this formula for a wide class of functions. We shall present the results of this investigation in form of a theorem.

T h e o r e m . Let $f(x)$ be a given function defined on the interval $(-\infty, \infty)$ and satisfying the following conditions.

1) Function $f(x)$ is piece-wise continuous and possesses a bounded variation in the open interval $(-\infty, \infty)$.

2) Further,

$$f(x)|x|^{-1/2}\ln(1+|x|)\in L(-\infty, -a)$$
$$f(x)x^{-1/2}\ln(1+x)\in L(a, \infty) \quad (a>0)$$

Then, we have

$$\frac{1}{2}[f(x+0)+f(x-0)] = \qquad (-\infty<x<\infty) \tag{1.3}$$

$$=\int_0^\infty \frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\Big\{\frac{P_{-1/2+i\tau}(ix)+P_{-1/2+i\tau}(-ix)}{2}\int_{-\infty}^{\infty} f(y)\frac{P_{-1/2+i\tau}(iy)+P_{-1/2+i\tau}(-iy)}{2}\,dy+$$
$$+\frac{P_{-1/2+i\tau}(ix)-P_{-1/2+i\tau}(-ix)}{2i}\int_{-\infty}^{\infty} f(y)\,\frac{P_{-1/2+i\tau}(iy)-P_{-1/2+i\tau}(-iy)}{2i}\,dy\Big\}\,d\tau$$

In particular, for the even function

$$\frac{1}{2}[f(x+0)+f(x-0)] = \qquad (-\infty<x<\infty) \tag{1.4}$$

$$=2\int_0^\infty \frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\,\frac{P_{-1/2+i\tau}(ix)+P_{-1/2+i\tau}(-ix)}{2}\,d\tau\int_0^\infty f(y)\,\frac{P_{-1/2+i\tau}(iy)+P_{-1/2+i\tau}(-iy)}{2}\,dy$$

while for the odd function

$$\frac{1}{2}[f(x+0)+f(x-0)] = \qquad (-\infty<x<\infty) \tag{1.5}$$

$$=2\int_0^\infty \frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\,\frac{P_{-1/2+i\tau}(ix)-P_{-1/2+i\tau}(-ix)}{2i}\,d\tau\int_0^\infty f(y)\,\frac{P_{-1/2+i\tau}(iy)-P_{-1/2+i\tau}(-iy)}{2i}\,dy$$

Last two formulas remain valid also for the functions defined on the interval $(0, \infty)$ and satisfying the conditions (*)

1) $f(x)$ is piece-wise continuous and possesses a bounded variation in the open interval $(0, \infty)$

2) $f(x)\in L(0, a), \qquad f(x)x^{-1/2}\ln(1+x)\in L(a, \infty), \qquad (a>0)$

2. Some estimates and asymptotic representations of spherical functions. We shall begin by establishing some necessary properties of spherical functions. From the definitions of Legendre functions $P_\nu(z)$ and $Q_\nu(z)$ ($Q_\nu(z)$ are

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*) These conditions do not presuppose that the function $f(x)$ is finite when $x = 0$.

functions of the second kind) it follows that, for any complex ν

$$\frac{P_{\nu-1/2}(i\sinh\alpha)+P_{\nu-1/2}(-i\sinh\alpha)}{2}=\frac{\Gamma(1/2+\nu)\cos(1/2\,\pi\nu-1/4\,\pi)}{\sqrt{2\pi}\,\Gamma(1+\nu)\sqrt{\cosh\alpha}}\times \tag{2.1}$$

$$\times\left\{e^{\alpha\nu}F\left(\frac{1}{2},\frac{1}{2},1+\nu,\frac{e^{\alpha}}{2\cosh\alpha}\right)+e^{-\alpha\nu}F\left(\frac{1}{2},\frac{1}{2},1+\nu,\frac{e^{-\alpha}}{2\cosh\alpha}\right)\right\}\qquad(-\infty<\alpha<\infty)$$

$$\frac{P_{\nu-1/2}(i\sinh\alpha)-P_{\nu-1/2}(-i\sinh\alpha)}{2i}=\frac{\Gamma(1/2+\nu)\sin(1/2\,\pi\nu-1/4\,\pi)}{\sqrt{2\pi}\,\Gamma(1+\nu)\sqrt{\cosh\alpha}}\times \tag{2.2}$$

$$\times\left\{e^{\alpha\nu}F\left(\frac{1}{2},\frac{1}{2},1+\nu,\frac{e^{\alpha}}{2\cosh\alpha}\right)-e^{-\alpha\nu}F\left(\frac{1}{2},\frac{1}{2},1+\nu,\frac{e^{-\alpha}}{2\cosh\alpha}\right)\right\}\qquad(-\infty<\alpha<\infty)$$

$$\frac{Q_{\nu-1/2}(i\sinh\alpha)+Q_{\nu-1/2}(-i\sinh\alpha)}{2}=-\left(\frac{\pi}{2}\right)^{1/2}\frac{\Gamma(1/2+\nu)\sin(1/2\,\pi\nu-1/4\,\pi)}{\Gamma(1+\nu)\sqrt{\cosh\alpha}}\times$$

$$\times\, e^{-\alpha\nu}F\left(\frac{1}{2},\frac{1}{2},1+\nu,\frac{e^{-\alpha}}{2\cosh\alpha}\right)\qquad(\alpha\geqslant0,\quad \nu\neq-2n-1/2,\ n=0,1,2,\ldots) \tag{2.3}$$

$$\frac{Q_{\nu-1/2}(i\sinh\alpha)-Q_{\nu-1/2}(-i\sinh\alpha)}{2i}=-\left(\frac{\pi}{2}\right)^{1/2}\frac{\Gamma(1/2+\nu)\cos(1/2\,\pi\nu-1/4\,\pi)}{\Gamma(1+\nu)\sqrt{\cosh\alpha}}\times$$

$$\times\, e^{-\alpha\nu}F\left(\frac{1}{2},\frac{1}{2},1+\nu,\frac{e^{-\alpha}}{2\cosh\alpha}\right)\qquad(\alpha\geqslant0,\ \nu\neq-2n-3/2,\ n=0,1,2,\ldots) \tag{2.4}$$

where $F(a, b, c, z)$ is a hypergeometric series

$$F(a,b,c,z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}z^k\qquad(|z|<1) \tag{2.5}$$

$$(\lambda)_k=\lambda(\lambda+1)\ldots(\lambda+k-1),\qquad(\lambda)_0=1$$

From (2.1) and (2.2), together with the inequality

$$|F(1/2,1/2,1+i\tau,x)|\leqslant F(1/2,1/2,1,x)\qquad(0\leqslant\tau<\infty,\ 0\leqslant x<1)$$

we easily obtain the estimates for the functions under consideration

$$\left|\frac{P_{-1/2+i\tau}(ix)+P_{-1/2+i\tau}(-ix)}{2}\right|\leqslant\left(\frac{\sinh\pi\tau}{\pi\tau}\right)^{1/2}\frac{P_{-1/2}(ix)+P_{-1/2}(-ix)}{2}$$

$$(0\leqslant\tau<\infty,\ -\infty<x<\infty) \tag{2.6}$$

$$\left|\frac{P_{-1/2+i\tau}(ix)-P_{-1/2+i\tau}(-ix)}{2i}\right|<\left(\frac{\sinh\pi\tau}{\pi\tau}\right)^{1/2}\frac{P_{-1/2}(ix)+P_{-1/2}(-ix)}{2}$$

$$(0\leqslant\tau<\infty,\ -\infty<x<\infty) \tag{2.7}$$

To obtain the asymptotic representations of spherical functions in the region $|\nu|\to\infty, |\arg\nu|\leqslant 1/2\,\pi$, we shall make use of the fact that every hypergeometric function present in (2.1) to (2.4) can be expressed in the form

$$F\left(\frac{1}{2},\frac{1}{2},1+\nu,x\right)=1+\sum_{k=1}^{\infty}\frac{(1/2)_k(1/2)_k}{(1+\nu)_k\,k!}x^k=$$

$$=1+\frac{x}{4(1+\nu)}\sum_{k=0}^{\infty}\frac{(3/2)_k(3/2)_k}{(2+\nu)_k(k+1)!}x^k=1+r(\nu,x),$$

where the estimate of the remainder (*) is

$$|r(\nu, x)| \leqslant \frac{x}{4|1+\nu|} \sum_{k=0}^{\infty} \frac{(^3/_2)_k}{k!} x^k = \frac{x}{4|1+\nu|}(1-x)^{-3/2} \tag{2.8}$$

From the above result,

$$F\left(\frac{1}{2}, \frac{1}{2}, 1+\nu, \frac{e^{\pm\alpha}}{2\cosh\alpha}\right) = 1 + e^{\pm 2\alpha}(e^{\pm 2\alpha}+1)^{1/2} O(|\nu|^{-1}) \qquad |\nu| \to \infty, \; |\arg\nu| \leqslant {}^1/_2\pi$$

follows, where O is independent of α. Using (2.8) together with known asymptotic formulas for the Gamma functions we find (**), that

$$\frac{P_{\nu-1/2}(i\sinh\alpha) + P_{\nu-1/2}(-i\sinh\alpha)}{2P_{\nu-1/2}(0)} = \frac{1}{2\sqrt{\cosh\alpha}}\{e^{\alpha\nu}[1+(e^{2\alpha}+1)^{3/2}O(|\nu|^{-1})] + e^{-\alpha\nu}[1+(e^{-2\alpha}+1)^{3/2}O(|\nu|^{-1})]\} \quad (-\infty<\alpha<\infty, |\nu|\to\infty, |\arg\nu| \leqslant {}^1/_2\pi) \tag{2.9}$$

$$\frac{\pi\nu P_{\nu-1/2}(0)}{\cos\pi\nu}\,\frac{P_{\nu-1/2}(i\sinh\alpha) - P_{\nu-1/2}(-i\sinh\alpha)}{2i} = \frac{1}{2\sqrt{\cosh\alpha}}\{e^{-\alpha\nu}[1+(e^{-2\alpha}+1)^{3/2}O(|\nu|^{-1})] - e^{\alpha\nu}[1+(e^{2\alpha}+1)^{3/2}O(|\nu|^{-1})]\} \quad (-\infty<\alpha<\infty, |\nu|\to\infty, |\arg\nu| \leqslant {}^1/_2\pi) \tag{2.10}$$

$$\frac{\nu P_{\nu-1/2}(0)}{\cos\pi\nu}\,\frac{Q_{\nu-1/2}(i\sinh\alpha) + Q_{\nu-1/2}(-i\sinh\alpha)}{2} = \frac{e^{-\alpha\nu}}{2\sqrt{\cosh\alpha}}[1+(e^{-2\alpha}+1)^{3/2}O(|\nu|^{-1})] \quad (\alpha \geqslant 0, |\nu|\to\infty, |\arg\nu| \leqslant {}^1/_2\pi) \tag{2.11}$$

$$\frac{Q_{\nu-1/2}(i\sinh\alpha) - Q_{\nu-1/2}(-i\sinh\alpha)}{2\pi i P_{\nu-1/2}(0)} = -\frac{e^{-\alpha\nu}}{2\sqrt{\cosh\alpha}}[1+(e^{-2\alpha}+1)^{3/2}O(|\nu|^{-1})] \quad (\alpha \geqslant 0, |\nu|\to\infty, |\arg\nu| \leqslant {}^1/_2\pi) \tag{2.12}$$

The above formulas differ form the usual asymptotic formulas for spherical functions, in that they contain the estimate of the remainder term, which is valid over the whole range of values of the variable α

3. Proof of the expansion theorem. The proof will be carried out separately for the odd and the even cases (Formulas (1.4) and (1.5)). The validity of the theorem for an arbitrary function $f(x)$ will then automatically follow from

$$f(x) = {}^1/_2[f(x) + f(-x)] + {}^1/_2[f(x) - f(-x)]$$

Let us assume that $f(x)$ is an even function and let us consider the integral

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*) Here we have used the inequality

$$\left|\frac{(^3/_2)_k}{(2+\nu)_k}\right|_{k=0,1,2,\ldots} \leqslant 1, \qquad |\arg\nu| \leqslant \frac{\pi}{2}$$

**) The multiplier

$$P_{\nu-1/2}(0) = \frac{\cos\pi\nu}{2\pi\sqrt{\pi}}\Gamma\left(\frac{1}{4}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{4}-\frac{\nu}{2}\right)$$

is introduced here in order to present the formulas (2.9) to (2.12) in a more symmetrical form.

$$J(T,x)=2\int_0^T \frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\frac{P_{-1/2+i\tau}(ix)+P_{-1/2+i\tau}(-ix)}{2}d\tau+$$
$$+\int_0^\infty f(y)\frac{P_{-1/2+i\tau}(iy)+P_{-1/2+i\tau}(-iy)}{2}dy \qquad (-\infty<x<\infty,\ T>0) \qquad (3.1)$$

The integrand represents here a continuous function of the parameter τ and a piece-wise continuous function y in the open interval $(0, \infty)$. From (2.6) we have the estimate (*)

$$\int_0^\infty \left|f(y)\frac{P_{-1/2+i\tau}(iy)+P_{-1/2+i\tau}(-iy)}{2}\right|dy\leqslant\left(\frac{\sinh\pi T}{\pi T}\right)^{1/2}\int_0^\infty |f(y)|\frac{P_{-1/2}(iy)+P_{-1/2}(-iy)}{2}dy=$$
$$=O(1)\left\{\int_0^a |f(y)|\,dy+\int_a^\infty |f(y)|\,y^{-1/2}\ln(1+y)\,dy\right\}$$

showing that the integral converges uniformly. Hence, the integral under consideration will be a continuous function of τ, and the repeated integral (3.1) will be meaningful. Further, uniform convergence allows us to change the order of integration, and write $J(T, x)$ as

$$J(T,x)=\int_0^\infty f(y)K(x,y,T)\,dy \qquad (3.2)$$

$$K(x,y,T)=2\int_0^T\frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\frac{P_{-1/2+i\tau}(ix)+P_{-1/2+i\tau}(-ix)}{2}\frac{P_{-1/2+i\tau}(iy)+P_{-1/2+i\tau}(-iy)}{2}d\tau \qquad (3.3)$$

Since the integrand is an even function of τ,

$$K(x,y,T)=-\frac{1}{i}\int_{-iT}^{iT}\frac{\nu\tan\pi\nu}{\cos\pi\nu}\frac{P_{\nu-1/2}(ix)+P_{\nu-1/2}(-ix)}{2}\frac{P_{\nu-1/2}(iy)+P_{\nu-1/2}(-iy)}{2}d\nu \qquad (3.4)$$

Let x be a fixed, positive number. Then, depending on whether $y\le x$ or $y\ge x$, we can represent the kernel $K(x, y, T)$ in one of the following forms (**) (3.5)

$$K(x,y,T)=\frac{2}{\pi i}\int_{-iT}^{iT}\frac{\nu}{\cos\pi\nu}\frac{P_{\nu-1/2}(iy)+P_{\nu-1/2}(-iy)}{2}\frac{Q_{\nu-1/2}(ix)+Q_{\nu-1/2}(-ix)}{2}d\nu \quad (y\leqslant x)$$

$$K(x,y,T)=\frac{2}{\pi i}\int_{-iT}^{iT}\frac{\nu}{\cos\pi\nu}\frac{P_{\nu-1/2}(ix)+P_{\nu-1/2}(-ix)}{2}\frac{Q_{\nu-1/2}(iy)+Q_{\nu-1/2}(-iy)}{2}d\nu \quad (y\geqslant x)$$

The integrands in (3.5) are analytical functions of ν, regular (***) on the semi-plane $\mathrm{Re}\,\nu\ge 0$. Hence, integration along the segment of the imaginary axis, can, in the following arguments, be replaced by integration along a semicircle Γ_T of radius T in the semi-plane $\mathrm{Re}\,\nu\ge 0$.

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*) $\dfrac{P_{-1/2}(iy)+P_{-1/2}(-iy)}{2}=\begin{cases}O(1), & y\in(0,a)\\ O(1)\,y^{-1/2}\ln(1+y), & y\in(a,\infty)\end{cases}$

**) A relation $\pi\tan\pi\nu P_{\nu-1/2}(z)=Q_{-\nu-1/2}(z)-Q_{\nu-1/2}(z)$ was taken into consideration in the process of this transformation.

***) Zeros of the denominator $\nu=1/2\,(2m+1)$, $m=0,1,2,\ldots$ cancel with the zeros of $Q_{\nu-1/2}(z)+Q_{\nu-1/2}(-z)$, when $m=2n$, and with the zeros of $P_{\nu-1/2}(z)+P_{\nu-1/2}(-z)$ when $m=2n+1$.

Let us investigate the integral $J(T, x)$ when $T \to \infty$. Assuming that $x = \sinh \alpha$, $y = \sinh \alpha'$, we have

$$J(T, x)=\int_0^\alpha f(\sinh\alpha') K(\sinh\alpha, \sinh\alpha', T)\cosh\alpha'\, d\alpha' + \int_\alpha^\infty f(\sinh\alpha') K(\sinh\alpha, \sinh\alpha', T)\cosh\alpha'\, d\alpha' =$$

$$= J_1(T, \sinh\alpha) + J_2(T, \sinh\alpha) \tag{3.6}$$

Using (2.9) and (2.11), we obtain for $\nu = Te^{i\varphi}\ (-{}^1/_2\pi \leqslant \varphi \leqslant {}^1/_2\pi)$.

$$\frac{\nu}{\cos\pi\nu}\,\frac{P_{\nu-1/2}(i\sinh\alpha') + P_{\nu-1/2}(-i\sinh\alpha')}{2}\,\frac{Q_{\nu-1/2}(i\sinh\alpha) + Q_{\nu-1/2}(-i\sinh\alpha)}{2} = \tag{3.7}$$

$$= \frac{1}{4\sqrt{\cosh\alpha\cosh\alpha'}}\left\{e^{-(\alpha-\alpha')\nu} + e^{-(\alpha+\alpha')\nu} + e^{-(\alpha-\alpha')T\cos\varphi}O(T^{-1}) + e^{-(\alpha+\alpha')T\cos\varphi}O(T^{-1})\right\}$$

while from (3.5) it follows that (*)

$$K(\sinh\alpha, \sinh\alpha', T) = \frac{1}{\sqrt{\cosh\alpha\cosh\alpha'}}\left\{\frac{\sin(\alpha-\alpha')T}{\pi(\alpha-\alpha')} + \frac{\sin(\alpha+\alpha')T}{\pi(\alpha+\alpha')} + \right.$$

$$\left. + O(1)\frac{1-e^{-(\alpha-\alpha')T}}{(\alpha-\alpha')T} + O(1)\frac{1-e^{-(\alpha+\alpha')T}}{(\alpha+\alpha')T}\right\} \qquad (0 \leqslant \alpha' \leqslant \alpha) \tag{3.8}$$

Substituting (3.8) into the first of the integrals of (3.6), we find that

$$J_1(T,\alpha) = \frac{1}{\pi}\int_0^\alpha f(\sinh\alpha')\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2}\frac{\sin(\alpha-\alpha')T}{\alpha-\alpha'}\,d\alpha' + \frac{1}{\pi}\int_0^\alpha f(\sinh\alpha')\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2}\frac{\sin(\alpha+\alpha')T}{\alpha+\alpha'}\,d\alpha' +$$

$$+ O(1)\int_0^\alpha |f(\sinh\alpha')|\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2}\frac{1-e^{-(\alpha-\alpha')T}}{(\alpha-\alpha')T}\,d\alpha' +$$

$$+ O(1)\int_0^\alpha |f(\sinh\alpha')|\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2}\frac{1-e^{-(\alpha+\alpha')T}}{(\alpha+\alpha')T}\,d\alpha' \tag{3.9}$$

According to the Dirichlet's theorem, we have, for $T \to \infty$ (**)

$$\frac{1}{\pi}\int_0^\alpha f(\operatorname{sh}\alpha')\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2}\frac{\sin(\alpha-\alpha')T}{\alpha-\alpha'}\,d\alpha' = \frac{1}{2}f(\sinh\alpha - 0) + o(1) \tag{3.10}$$

$$\frac{1}{\pi}\int_0^\alpha f(\sinh\alpha')\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2}\frac{\sin(\alpha+\alpha')T}{\alpha+\alpha'}\,d\alpha' = o(1) \tag{3.11}$$

Further, if the interval of integration is divided into the subintervals $(0, \alpha - \delta)$ and $(\alpha - \delta, \alpha)$ and if a sufficiently small positive δ (implying a sufficiently large T) is chosen, we have

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*) Here we utilize a known inequality

$$\int_0^{1/2\pi} e^{-\lambda T\cos\varphi}\,d\varphi \leqslant \frac{\pi}{2}\,\frac{1-e^{-\lambda T}}{\lambda T} \qquad (\lambda \geqslant 0)$$

after transforming the first of the integrals of (3.5) into the line integral along the arc Γ_T .

**) Conditions imposed on $f(x)$ imply, that $f(\sinh\alpha')\sqrt{\cosh\alpha'} \in L(0, \infty)$.

$$\int_0^{\alpha} |f(\sinh\alpha')| \left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2} \frac{1-e^{-(\alpha-\alpha')T}}{(\alpha-\alpha')T}\, d\alpha' \leqslant \frac{1}{\delta T} \int_0^{\alpha-\delta} |f(\sinh\alpha')| \sqrt{\cosh\alpha'}\, d\alpha' +$$

$$+ \int_{\alpha-\delta}^{\alpha} |f(\sinh\alpha')| \sqrt{\cosh\alpha'}\, d\alpha' = O(T^{-1}) + o(1) = o(1) \qquad T\to\infty \tag{3.12}$$

and finally

$$\int_0^{\alpha} |f(\sinh\alpha')| \left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2} \frac{1-e^{-(\alpha+\alpha')T}}{(\alpha+\alpha')T}\, d\alpha' \leqslant \frac{1}{\alpha T} \int_0^{\alpha} |f(\sinh\alpha')| \sqrt{\cosh\alpha'}\, d\alpha' =$$

$$= O(T^{-1}) = o(1), \qquad T\to\infty \tag{3.13}$$

(3.9) to (3.13) leads to

$$\lim_{T\to\infty} J_1(T, \sinh\alpha) = {}^1/_2 f(\sinh\alpha - 0) \tag{3.14}$$

Investigation of the integral J_2 is performed in a similar manner and it leads to

$$J_2(T,\sinh\alpha) = \frac{1}{\pi}\int_{\alpha}^{\infty} f(\sinh\alpha')\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2} \frac{\sin(\alpha'-\alpha)T}{\alpha'-\alpha}\, d\alpha' +$$

$$+ \frac{1}{\pi}\int_{\alpha}^{\infty} f(\sinh\alpha')\left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2} \frac{\sin(\alpha'+\alpha)T}{\alpha'+\alpha}\, d\alpha' +$$

$$+ O(1)\int_{\alpha}^{\infty} |f(\sinh\alpha')| \left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2} \frac{1-e^{-(\alpha'-\alpha)T}}{(\alpha'-\alpha)T}\, d\alpha' +$$

$$+ O(1)\int_{\alpha}^{\infty} |f(\sinh\alpha')| \left(\frac{\cosh\alpha'}{\cosh\alpha}\right)^{1/2} \frac{1-e^{-(\alpha'+\alpha)T}}{(\alpha'+\alpha)T}\, d\alpha' \tag{3.15}$$

From this we find, as above

$$\lim_{T\to\infty} J_2(T,\sinh\alpha) = {}^1/_2 f(\sinh\alpha + 0) \tag{3.16}$$

and hence

$$\lim_{T\to\infty} J(T, x) = {}^1/_2 [f(x+0) + f(x-0)] \tag{3.17}$$

We have proved (3.17) for positive x. Its validity for $x < 0$ follows from the fact that the functions $J(T, x)$ and $f(x)$ are even functions of x. Expression (3.17) can also be shown to be valid for $x = 0$, by changing slightly the course of reasoning.

Thus we have proved the expansion theorem (1.4) for the even case. For the odd case the proof is analogous. Formulas (2.6),(2.9) and (2.11) however, must be replaced by (2.7),(2.10) and (2.12), respectively.

4. Examples. (1) Let

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases} \tag{4.1}$$

Using the identity

$$\frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} = -\frac{1}{{}^1/_4+\tau^2}\frac{d}{dx}\sqrt{x^2+1}\,\frac{P^1_{-1/2+i\tau}(ix) + P^1_{-1/2+i\tau}(-ix)}{2} \tag{4.2}$$

where $(P^1_\nu(z)$ is an adjoint Legendre function, and applying Theorem (1.4), we obtain

$$f(x) = -2\sqrt{a^2+1} \times \qquad (-\infty < x < \infty) \tag{4.3}$$

$$\times \int_0^\infty \frac{\tau \tanh \pi\tau}{(1/4+\tau^2)\cosh \pi\tau} \, \frac{P^1_{-1/2+i\tau}(ia) + P^1_{-1/2+i\tau}(-ia)}{2} \, \frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} \, d\tau$$

2) Assuming $f(x) = (x^2+1)^{-1/2}$ and taking into account

$$\int_0^\infty \frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} \, \frac{dx}{\sqrt{x^2+1}} = \frac{\pi}{\cosh \pi\tau} \{P_{-1/2+i\tau}(0)\}^2 \tag{4.4}$$

we find that

$$\frac{1}{\sqrt{x^2+1}} = 2\pi \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2 \pi\tau} \{P_{-1/2+i\tau}(0)\}^2 \, \frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} \, d\tau$$

$$(-\infty < x < \infty) \tag{4.5}$$

3) The following expansion represents a generalization of (4.5)

$$\frac{1}{\sqrt{x^2+a^2+1}} = \qquad (-\infty < x < \infty) \tag{4.6}$$

$$= 2\pi \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2 \pi\tau} P_{-1/2+i\tau}(0) \, \frac{P_{-1/2+i\tau}(ia) + P_{-1/2+i\tau}(-ia)}{2} \, \frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} \, d\tau$$

4) In conclusion we shall give an example of expansion of a function possessing an infinite discontinuity

$$f(x) = \begin{cases} (a^2 - x^2)^{-1/2} & |x| < a \\ 0 & , \ |x| > a \end{cases} \tag{4.7}$$

$$f(x) = \pi \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh \pi\tau} P_{-1/2+i\tau}(0) \, P_{-1/2+i\tau}\left(\sqrt{a^2+1}\right) \frac{P_{-1/2+i\tau}(ix) + P_{-1/2+i\tau}(-ix)}{2} \, d\tau \tag{4.8}$$

Although the assumption of piece-wise continuity of the function is violated in this case at the points $x = \pm a$, the expansion theorems (1.3) to (1.5) remain valid.

BIBLIOGRAPHY

1. Mehler, F.G., Über eine mit Kügel und Cylinder Funktionen verwandte Funktion und ihre Anwendung in der Theorie der Elektricitäts-verteilung. Math,Annln., Vol.18, 1881.
2. Fok, V.A., O razlozhenii proizvol'noi funktsii v integral po funktsiiam Lezhandra s kompleksnym znachkom (On the integral expansion of an arbitrary function in terms of Legendre functions with a complex index). Dokl.Akad.Nauk SSSR, Vol.39, № 7, p.279, 1943.
3. Lebedev, N.N., Spetsial'nye funktsii i ikh prilozheniia (Special Functions and their Applications). Izd.M., Fizmatgiz, 2nd edition, 1963.
4. Lebedev, N.N., Reshenie problemy Dirikhle dlia giperboloidov vrashcheniia (Solution of the Dirichlet problem for hyperboloids of revolution). *PMM* Vol.11, № 2, p.251, 1947.
5. Lebedev, N.N., Skal'skaia, I.P. and Ufliand, Ia.S., Sb. zadach po matem. fiz. (Collected Problems of Mathematical Physics). M, Gostekhizdat,1955.
6. Ufliand, Ia.S., Integral'nye preobrazovaniia v zadachakh teorii uprugosti (Integral Transformations in Problems of the Theory of Elasticity). M.-L., Izd.Akad.Nauk SSSR, 1963.
7. Weyl, H., Über gewöhnliche lineare Differential-gleichungen mit singulären Stellen und ihre Eigenfunktionen. Nachr.Königl.Gesellschaft Wissenschaften Göttingen, 1910.
8. Titchmarsh, E.Ch., Razlozheniia po sobstvennym funktsiiam, sviazannye s differentsial'nymi uravneniiami vtorogo poriadka (Expansions in terms of characteristic functions, connected with differential equations of second order). M, Izd.inostr.Lit., 1960.

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